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Fractals, Dimensional Analysis and Similarity, and Turbulence

*Paul E. Dimotakis**

[Paul Dimotakis and K. R. Sreenivasan were invited to each write a brief piece on the relevance of fractals to the analysis and description of turbulence. The first of these follows: we hope to publish the second in a subsequent issue.—THE EDITORS]

In a series of papers and two books, Mandelbrot proposed to apply the notion of stochastic geometric similarity to describe a host of physical phenomena. This description was proposed as appropriate to phenomena that do not possess an inherent characteristic scale and give rise to sets that can be covered by a number $N(\lambda)$ of elements of size λ, given by

$$N(\lambda) \propto \lambda^{-D}. \qquad (1)$$

Sets whose coverage could be described in this manner he dubbed *fractal*, with the exponent D identified as the associated *fractal dimension*. See discussion in Mandelbrot [1977, 1982], and references therein.

Mandelbrot proposes that, in practice, the fractal dimension D can be determined by a straight line fit to a log–log plot of $N(\lambda)$ versus λ of the coverage law of Eq. (1). Equivalently, it can be computed from the logarithmic derivative

$$D = -\frac{\mathrm{d}\log N(\lambda)}{\mathrm{d}\log\lambda}. \qquad (2)$$

Strictly speaking, the dimension D is implicitly defined by Eq. (1), or Eq. (2), in the limit of $\lambda \to 0$. This is also the case for the *Hausdorff–Besicovitch dimension*, defined as the critical value of the exponent D, where the measure

$$\mathcal{M}(\lambda; D) \equiv \sum_{\mathcal{S}} \gamma(D)\lambda^{D} = \gamma(D)N(\lambda)\lambda^{D} \qquad (3)$$

transitions from zero to infinity, in the limit of $\lambda \to 0$. The summation is to be taken over the set $\mathcal{S}$ with a factor $\gamma(D)$ that

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Short time exposure photograph of burning, three-dimensional surface in a turbulent diffusion flame. Are such turbulent, flow-generated, interfacial surfaces fractal? (Photograph © 1991 by Paul E. Dimotakis.)

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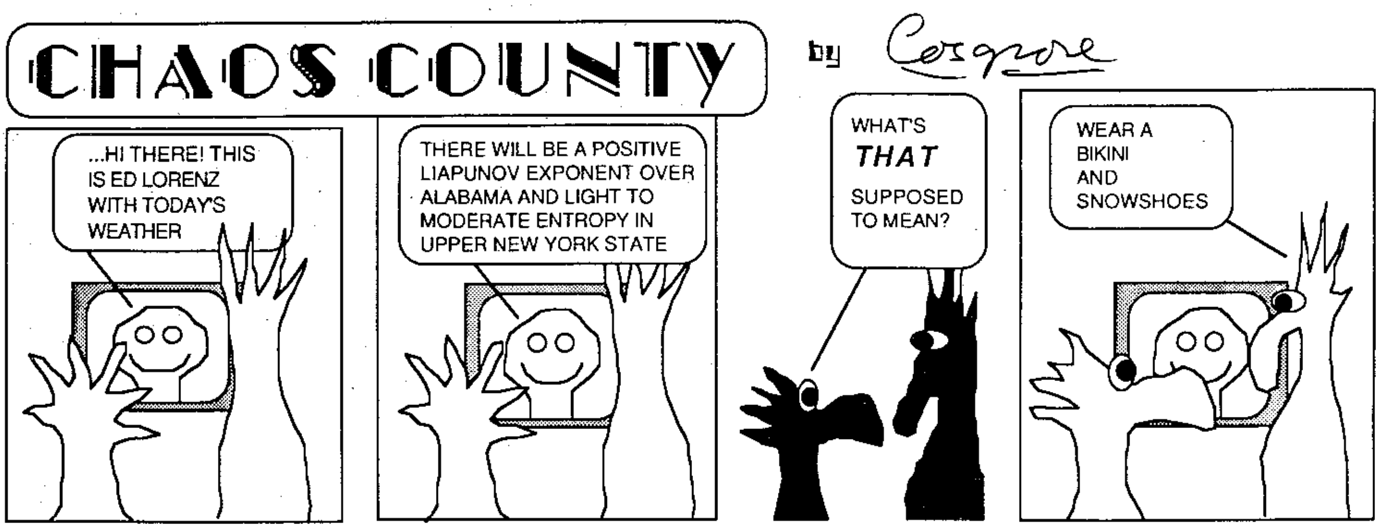

Dimotakis *Continued from p. 1*

can depend on the elements used to cover the set (see, for example, Feder [1988, p. 14]). In the Mandelbrot 1977 and 1982 discussions, the two quantities are used interchangeably, by considering cases, as he says, for which this equivalence can be accepted.

As Mandelbrot notes [1982, pp. 357–360], the basic mathematical ideas can be traced to earlier proposals and discussions. Equation (2) was essentially used by Pontrjagin and Schnirelman in the 30s, and the quantity D defined by Eqs. (1) and (2) can be identified with the *Kolmogorov capacity* (Kolmogorov and Tihomirov, 1959), or, as it was subsequently dubbed, the *box-counting* dimension. There can be no doubt, however, that it was Mandelbrot who realized the importance of these ideas, brought them to the attention of a community outside that of pure mathematics, and argued that they should be expected to prove applicable and useful in the description of many physical phenomena. A considerable body of research has appeared on the record in response to his proposals relating to a spectrum of phenomena too numerous to trace.

In the context of many of the phenomena proposed by Mandelbrot to be potentially describable by the fractal power law coverage similarity, however, one must require that Eqs. (1) and (2) hold for λ not only strictly positive, but over some nonvanishing range of scales λ, i.e., for

$$0 < \lambda_{\min} \leq \lambda \leq \lambda_{\max}. \tag{4}$$

This is a necessary assumption if one is to derive many of the benefits of this analysis. It will also prove important on another score.

Specifically, to return to the basic proposal, it is important to note that as regards the description of physical entities in which λ has units, Eq. (1) *is not dimensionally correct;* only a dimensionless group can be equal to a pure number. Of course, one could argue that a proportionality constant C is implicit in Eq. (1), i.e.,

$$N(\lambda) = C\lambda^{-D}. \tag{5}$$

Such a constant would be *dimensional,* however, with units as dictated by those of λ and the value of D, i.e., $C = \mathrm{fn}(D)$. In using Eq. (5) as an implicit definition of D, in other words, in which D is the independent variable, the "constant" of proportionality C is not a constant at all and hides important scaling information.

To illustrate the point, consider a set representing a surface in a d-dimensional (Euclidean) space, for example, fixed threshold crossing of a scalar time trace, or linear image ($d = 1$); isoscalar contours of a scalar field, e.g., $c(x,y) = c_{\text{th}}$, in a planar image ($d = 2$); or isoscalar surfaces, e.g., $c(x,y,z) = c_{\text{th}}$, of a scalar field in three dimensions ($d = 3$). Consider a volume given by L^d, in each case, where L is the (linear) extent of the space spanned by the data. The total number of λ-sized elements that can fit in this space is given by $(L/\lambda)^d$.† The total coverage $N_d(\lambda)$ of the set can then be expressed as the fraction of the number of λ-elements covering the set, times the total number of λ-elements in the d-dimensional (data) space, i.e.,

$$N_d(\lambda) = F_d(\lambda)\left(\frac{L}{\lambda}\right)^d. \tag{6}$$

The proposal that $N_d(\lambda) \propto \lambda^{-D_d}$ is then equivalent to a dependence of the coverage fraction on λ, i.e., the probability that a λ-element covers a portion of the set, given by

$$F_d(\lambda) \propto \lambda^{d-D_d}. \tag{7}$$

Since $F_d(\lambda)$ is a probability and therefore dimensionless, Eq. 7 requires that λ must be scaled by some like-dimensioned quantity, or characteristic scale, λ_0, i.e.,

$$F_d(\lambda) = F_{d0}\left(\frac{\lambda}{\lambda_0}\right)^{d-D_d} \tag{8}$$

We can combine this with Eq. 6 to yield what is implicit in the original equation, namely,

$$N_d(\lambda) = F_{d0}\left(\frac{\lambda}{\lambda_0}\right)^{d-D_d}\left(\frac{L}{\lambda}\right)^d. \tag{9}$$

In this expression, the now dimensionless constant F_{d0} is equal to the required fraction of set-covering elements of size $\lambda = \lambda_0$, provided, of course, the characteristic scale λ_0 falls within the $(\lambda_{\min},\lambda_{\max})$ interval in which Eq. 9 applies. While, in principle, the factor F_{d0} could be absorbed in a redefined λ_0, it is useful to retain it; the characteristic scale λ_0 might well be definable on some other physical basis quite independently of a fit to the coverage of the set of interest using Eq. 9.

†Barring edge effects, which, however, can be eliminated in any of a number of ways—for example, by considering λ's that are exact binary submultiples of L, i.e., $\lambda_n = L/2^n$.

As an aside, it might appear that Eq. (3), the Hausdorff–Besicovitch expression for D, is not subject to the same difficulty. Strictly speaking, it is dimensionally correct as it stands. Nevertheless, one would prefer not to be using a measure that is *dimensional* (with units that depend on the value of D), or to be taking the limit with a *dimensional* $\lambda \to 0$, to determine the critical value of D. Instead, one would prefer a measure based on a *scaled* λ, e.g., $\mathcal{M}(\lambda/\lambda_0; D)$, with D computed as the critical exponent in the limit of $\lambda/\lambda_0 \to 0$. Again, in other words, there is a need for a characteristic scale λ_0, even if not as manifestly as with the box-counting coverage expression of Eq. (1).

These observations have an interesting consequence.

Returning to Eqs. (1), (5), or (9), we see that they can be expected only to apply to processes that observe a characteristic scale λ_0. The resulting coverage function $N(\lambda)$ could then depend strongly on this characteristic scale, or not, depending on the value of the dimension exponent D. More importantly, however, the coverage of sets generated by processes that do not possess such a characteristic scale cannot be expected to be described by these equations.

Turning to the more specific question, an area in which there have been many attempts to apply power law fractals is ". . . the grand chapter of Physics, the study of turbulence in fluids" [Mandelbrot, 1977, p. 145]—the dynamics, in other words, of high Reynolds number fluid motion. Turbulent-flow-generated sets, especially at high Reynolds numbers, are known to exhibit a host of similarity properties. These can be expected to hold over a large range of space and time scales that are bracketed by

$$\lambda_{max} \sim \delta, \tag{10a}$$

the outer flow scale, and

$$\lambda_{min} \sim \lambda_K \equiv \left(\frac{\nu^3}{\bar{\epsilon}}\right)^{1/4}, \tag{10b}$$

the inner Kolmogorov [1941a] scale, in which ν is the kinematic viscosity and $\bar{\epsilon}$ is the mean dissipation rate of the kinetic energy per unit mass. At high Reynolds numbers Re, the bounds of this (inertial) range are in the ratio

$$\frac{\lambda_{max}}{\lambda_{min}} \sim \frac{\delta}{\lambda_K} \sim \mathrm{Re}^{3/4}. \tag{11}$$

This potentially very large similarity range would seem to provide a promising arena for the Mandelbrot fractal proposals. Sets generated by turbulent flow processes that have been analyzed in this light include selected velocity component time traces, estimates of the local energy dissipation and its fluctuations, temporal as well as spatial scalar fluctuations and isoscalar surfaces, etc. The recent review by Sreenivasan [1991] can be consulted for a discussion and references.

Let us examine this proposal in view of some of the similarity properties in the inertial range of scales and the preceding discussion.

In his first and second similarity hypotheses, Kolmogorov [1941a] proposed that the only dynamically important quantity at a scale $\lambda = 2\pi/k$ within the range of Eq. (4), with the identifications in Eqs. (10), is the mean dissipation rate $\bar{\epsilon}$. This proposal immediately leads to the celebrated Kolmogorov $-5/3$ spectrum by dimensional analysis, i.e.,

$$E(k) \simeq C_K \bar{\epsilon}^{2/3} k^{-5/3}, \tag{12a}$$

where C_K is a (dimensionless) constant of proportionality of order unity. The right-hand side is the only group that can be formed with $\bar{\epsilon}$ and k that has the requisite dimensions.

Physically, the first and second Kolmogorov hypotheses are equivalent to assuming that the dynamics of the turbulent straining motions that take an eddy of a scale λ_1 to a scale λ_2, for

$$\delta \gg \lambda_1 > \lambda_2 \gg \lambda_K,$$

depend only on the local scale and therefore only on the ratio λ_1/λ_2. In particular, they are equivalent to assuming that the dynamics do not depend on the ratio of the local scale λ to the outer scale δ, the inner scale λ_K *or any other scale*.

In subsequent refinements by Kolmogorov [1962], Oboukhov [1962], and others, the kinetic energy dissipation rate $\epsilon = \epsilon(\mathbf{x},t)$ was accommodated as the local, intermittent field that it is. This introduced the outer scale δ as a scale *weakly* admissible in the dynamics, resulting in small corrections to the $-5/3$ spectrum. Specifically, with a log-normally distributed $\epsilon(\mathbf{x},t)$, the energy spectrum becomes

$$E(k) \simeq C_K \bar{\epsilon}^{2/3} (k\delta)^{-\mu/9} k^{-5/3}, \tag{12b}$$

where μ is the intermittency exponent, estimated to be in the range of $0.2 < \mu < 0.5$ (e.g., Monin and Yaglom [1975, p. 642]). Significantly, the difference between Eqs. (12a) and (12b) is too small to be discernible experimentally. At least by this measure, we may conclude that the Kolmogorov [1941a] hypothesis of scale independence, that leads to the $-5/3$ spectrum law, must very nearly be right.

This conclusion has consequences, in light of the preceding discussion of the need for a characteristic scale to exist, if the power law fractal similarity is to describe the coverage. In the case of sets with scales in the inertial range and the (near) absence of a characteristic scale λ_0, we cannot expect Eq. (9) to apply if the fractal dimension exponent D_3 for coverage of three-dimensional sets is to be assigned values in the range of $D_3 = 2.36 \pm 0.05$, as was suggested, for example, by Sreenivasan [1991, p. 556]. This is also a value close to $D_3 = 8/3$ that Mandelbrot suggests "is reducible to the Kolmogorov theory and the empirical spectra,"‡ which he also argues corresponds to a value of $D_2 = 5/3$ for planar sections [Mandelbrot 1977, p. 52].

We note here that, implicit in the assignment of both the reported experimental estimate of $D_3 = 2.36 \pm 0.05$ of Sreenivasan, as well as the correspondence between the value of $D_3 = 8/3$ and $D_2 = 5/3$ proposed by Mandelbrot as applicable to isoscalar surfaces, is the relation

$$d_1 - D_{d_1} = d_2 - D_{d_2}, \tag{13}$$

where d_1 and d_2 denote the topological dimension, e.g., $D_3 = D_2 + 1$ and $D_3 = D_1 + 2$, as was asserted by Sreenivasan and Meneveau [1986, Sec. 2.2]—recall also Eqs. (8) and (9). Specifically, in the case of the experimental values reported, the value of D_3 cited was inferred from measurements of one-dimensional ($d = 1$) or two-dimensional ($d = 2$) data, assuming Eq. (13). A suggestion to that effect had previously been made by Mandelbrot, whom Sreenivasan and Meneveau [1986] cite as a reference for this, and who says that "it is 'almost sure' " that the intersection of a three-dimensional surface with a plane yields a dimension $D_2 = D_3 - 1$ [Mandelbrot, 1982, p. 135], especially if the three-dimensional surface of interest is irregular (ibid., p. 136). The latter proviso would of course be satisfied in the case for an isoscalar surface in turbulence, for example (see also the discussion in Mandelbrot [1976, p. 124; 1977, p. 52] cited above). We will return to this point below.

Accepting Eq. (13) at face value for now, we can argue that such values for D_3 for an isoscalar surface, for example, are too far from the topological dimension of a three-dimensional object, i.e., $D_3 = 3$, and, despite Mandelbrot's assertion, that the resulting dependence of the coverage function $N_3(\lambda)$ on whatever

‡It is not clear whether Mandelbrot, in his reference here to "the Kolmogorov theory and the empirical spectra," is thinking of the original (1941a) arguments leading to the $-5/3$ spectrum [Eq. (12a), or the revised (1962) proposals that lead to Eq. (12b)].

characteristic scale λ_0 is used in Eq. (9) is too strong to be consistent with turbulent flow inertial range similarity. This conclusion can be reached by considering the properties of an isoscalar surface and its proposed (power law) fractal coverage $N_3(\lambda)$, assuming that

$$\lambda_0 \propto \delta, \tag{14}$$

as one might in view of the turbulent cascade process from larger to smaller scales. If we consider, for example, two gedanken experiments in the same type of flow, arranged to generate the same inner scales λ_K, but with outer scales δ_1 and δ_2 differing by, say, a factor of 10, and compare the coverage $N_3(\lambda)$ at a fixed scale λ, say, for $\lambda = 10\lambda_K \ll \delta_1, \delta_2$. The analysis of such a comparison is beyond the purposes of the present brief, informal discussion and is not included here.

This argument was recently noted in Miller and Dimotakis [1991], who did not find Eq. (1) [or, equivalently, Eq. (9)] applicable to measurements of temporal isoscalar intersections at a fixed point, or to spatial measurements derived from linear images, in the far field of turbulent jets where it had previously been reported to apply (e.g., Sreenivasan and Meneveau [1986]). In a note added in proof, Sreenivasan [1991] suggested that the discrepancy could be attributable to the fact that Miller and Dimotakis analyzed temporal data. As noted above, however, not only had such behavior been reported previously for temporal data,[#] but in their discussion Miller and Dimotakis included measurements and the analysis of temporal, spatial, as well as spatio-temporal (streak image) data, with similar results.

Briefly, to cover a total of N_T transitions in a one-dimensional record of length L, we must find

$$N_1(\lambda) \to \begin{cases} N_T & \text{for } \lambda \to 0 \\ L/\lambda & \text{for } \lambda > \delta. \end{cases} \tag{15}$$

The small-scale limiting behavior can be understood by appreciating that, once the covering element scale λ has decreased below the shortest separation between threshold transitions in the scalar trace, the total number of elements of size λ required to cover those transitions becomes equal to the total number N_T of transitions in the record of length L.

The large-scale limiting behavior can be understood by appreciating that *every* covering element of size λ much larger than the outer scale δ will cover transitions and, therefore, the total number of elements required to cover those transitions must tend to the total number of elements L/λ that can fit in the record. In other words, we must have $F_1(\lambda) \to 1$ for $\lambda \gg \delta$ (cf. Eq. 6). It is also easy to appreciate that, for typical traces derived from turbulent flow, the number of elements $N_1(\lambda)$ needed to cover the corresponding sets would only gradually approach the limiting values in Eq. (15), for small and large λ.

In view of the above and Eq. (2), we must have, for one-dimensional isoscalar data, that

$$D_1 \to \begin{cases} 0 & \text{for } \lambda < \lambda_B \\ 1 & \text{for } \lambda > \delta, \end{cases} \tag{16}$$

where λ_B is the Batchelor [1959] scalar diffusion scale (below which the isoscalar surface can be regarded as differentiable), and that, therefore, the exponent D_1 cannot be treated as a constant.

While it may be a minor point, it should be noted that if the measurements are alias-free with some margin, i.e., if they have been sampled at a frequency comfortably higher than the Nyquist frequency, and sufficiently noise-free, $N(\lambda)$ should tend to a constant at the smallest values of λ resolved by the data [for the reasons stated in reference to Eq. (15)] whether or not the smallest scale of the physical process has been resolved. It should also tend to the corresponding limit at large λ if the record length L spans a sufficient number of outer flow scales. Accordingly, we can always expect the limits in Eq. (16) to be attained by the analysis of proper data.

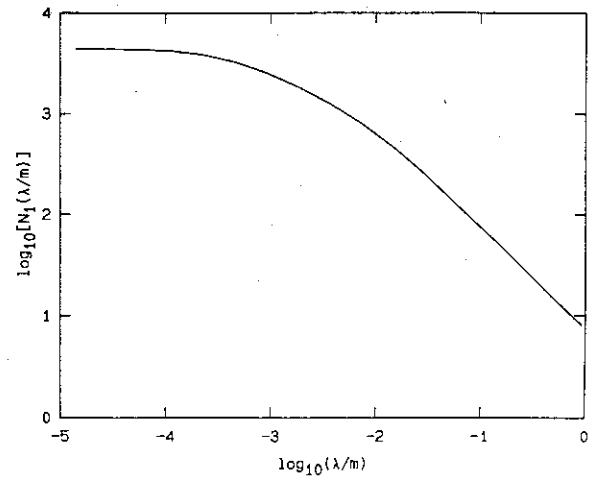

Figure 1: Coverage $N_1(\lambda)$ derived from temporal data of jet fluid concentration crossings of the local $\bar{c}$, measured on the axis of a turbulent jet at $x/d = 100$ and a Reynolds number of 1.15×10^4. The scale λ here is converted to meters. Data from Miller and Dimotakis [1991].

Of course, the preceding alone does not preclude, *a priori*, the existence of a $D_1(\lambda)$ that osculates a constant value over a range of scales λ,

$$\lambda_B < \lambda_{min} < \lambda < \lambda_{max} < \delta,$$

in the case of isoscalar surfaces. That is rendered unlikely by the additional requirement for a characteristic scale λ_0 appropriate to that range, as noted above.

Rewriting Eq. (2), we conclude that, in general, we must allow the exponent D to be a function of λ, i.e.,

$$D_d(\lambda) = -\frac{d \log N_d(\lambda)}{d \log \lambda}, \tag{17}$$

as suggested in Miller and Dimotakis. In the analysis of their turbulent jet fluid scalar data, they find a $D_1(\lambda)$ smoothly rising from 0 to 1, over a range of jet Reynolds numbers, $3.0 \times 10^3 \leq Re \leq 2.4 \times 10^4$, with no indication of any preferred intermediate value. They also report that the observed $D_1(\lambda)$ is very close to one derived from a simple model process in which scalar threshold transitions are log-normally spaced.

Figure 1 depicts $N_1(\lambda)$, as required to cover transitions of the scalar jet fluid concentration trace $c(t)$, of a threshold set at the local mean $\bar{c}$,* on the axis of a turbulent jet in water, at $x/d = 100$, and a Reynolds number of 1.15×10^4. In this plot, the covering element scale λ was converted to an equivalent spatial extent, measured in meters, to facilitate comparison with other temporal data, at both higher and lower Reynolds numbers (cf. Miller and Dimotakis [1991, Fig. 4]). The outer scale (local jet diameter) is estimated to be $\delta \simeq 0.11$ m, i.e., $\log_{10}(\delta/m) \simeq$

[#]Note reference to the use of Taylor's frozen flow hypothesis in Sreenivasan, Ramshankar and Meneveau, [1989, Table 1], Note (d), where the Sreenivasan and Meneveau [1986] jet data are cited.

*The important issue of the choice of the threshold was addressed in Miller and Dimotakis [1991]. The reader is referred to that discussion for further details.

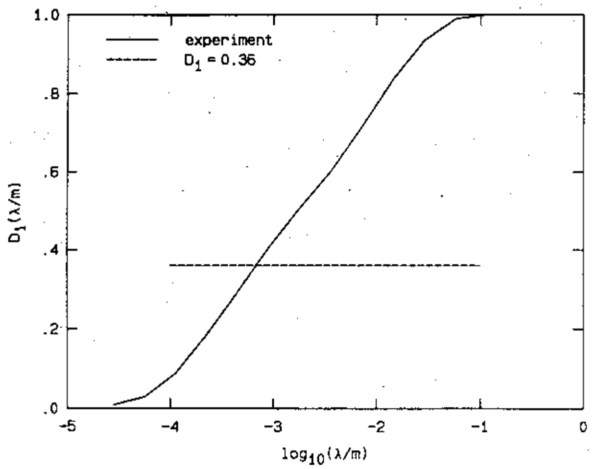

Figure 2: Logarithmic derivative $D_1(\lambda/m)$, Eq. (17), of the coverage $N_1(\lambda)$ plotted in Figure 1.

-1.0. As can be seen, the computed coverage observes the required limits noted in Eq. (15).

Figure 2 plots the logarithmic derivative $D_1(\lambda)$, computed from the $N_1(\lambda)$ coverage plotted in Figure 1. Note that the measured $D_1(\lambda) \rightarrow 0$, at the smallest scales, and $D_1(\lambda) \rightarrow 1$ for $\lambda \sim \delta$, or larger, as required. The dashed line in Figure 2 marks the level $D_1 = 0.36$. The latter was the value reported to have been derived from direct measurements of one-dimensional temporal data by Sreenivasan and Meneveau [1986]. It would also be the value derived from the previously cited estimate of $D_3 \simeq 2.36$ if the assertion that $D_3 = D_1 + 2$ is accepted [Eq. (13)]. However, while the assertion of Eq. (13) may be true in special cases, it must in general be false, as will be argued below.

Consider an isotropic random surface in three dimensions, which we accept as possessing a fractal dimension D_3, and its intersection with a plane, yielding a two-dimensional curve, which we will also accept as possessing a fractal dimension D_2. Using the two-dimensional curve, we generate (extrude) a cylindrical surface by translating the two-dimensional curve along a direction perpendicular to its plane. As Feder [1988, Fig. 13.1 and related discussion] notes, the dimension of the resulting cylindrical surface, which has very different space-filling properties, is given by $D_2 + 1$. Equivalently, we can expect the required coverage fraction $F_3(\lambda)$, cf. Eqs. 6, 8, and 9, to be different for the original three-dimensional isotropic surface, vs. the generated cylindrical surface.

This suggests that for a random surface in three-dimensions, of dimension D_3, and a planar section of the same surface of dimension D_2, we can expect, in general,

$$D_3 \neq D_2 + 1 \tag{18}$$

(analogously, for curves that are originally planar of dimension D_2 versus linear cuts yielding intersections of dimension D_1).

Is the implication that there can be no (power law) fractals? Not at all.

To return to the earlier point on similarity and dimensional analysis, the inference that should be drawn is that a (power law) fractal coverage relation requires a characteristic scale. Without that, and as opposed to the original proposals, we cannot write Eq. (1) dimensionally correctly. Conversely, processes that do possess relevant characteristic scales could exhibit (power law) fractal behavior.

What is then the relevant characteristic scale for the triadic Cantor set (e.g., Mandelbrot [1977, p. 98]), often described as self-similar, for example?

It is easy to appreciate that, for the triadic Cantor set, the characteristic scale is the original length L that is successively subdivided into thirds, i.e., $\lambda_0 = L$ in Eq. 9. The resulting set is obviously *not* self-similar at all scalings; it is not the same if we cut the original length L in half, for example. It is also not self-similar homogeneously; successive subdivisions do not endow the middle third of the original L with any Cantor "dust." Only the first and last third subsegments of any (sub)segment of the Cantor set are similar. This represents a rather restricted similarity scaling, which is pegged to the original length L and a coverage phased with respect to the original segment. The oscillations in the calculations of Smith et al. [1986], as well as in the spectrum of the triadic Cantor set,† can be understood in this light.

Accepting that the geometry of coastlines may be describable by a power law fractal relation, for example (cf. Feder [1988, Fig. 2.7]), one can think of several candidate characteristic scales that could be relevant. Potentially, these range from buoyant scales of the motion within the earth's mantle, responsible for tectonic motion on continental dimensions, to ocean wave scales at intermediate topography scales, to rock grain boundary and fracture scales, and below, as appropriate for microscopic and submicroscopic stochastic geometry scales. One could imagine the application of Eq. (9), corresponding to each of these separate characteristic scales, with, potentially, a different value of the exponent D for each one.

In the context of fluid mechanics and turbulence, this is also not to say that (power law) fractal relations and constant fractal dimension exponents are ruled out. The arguments here attempt to address issues arising in fully developed, turbulent flow in the intermediate, inertial range of scales, where power law fractals have been proposed to apply. Conversely, however, isoscalar surfaces that, for example, demark the boundaries of a turbulent region and even may serve as the definition of the outer scale δ are susceptible to dynamics that are obviously not independent of a characteristic scale, namely, the outer scale δ. Such interfaces would then appear to possess the necessary condition for a power law fractal description to be applicable and may indeed be found to display such behavior. It would then, however, be surprising if the resulting fractal dimension exponents were found to be universal, i.e., independent of the details of the flow geometry, Reynolds number, Schmidt number, etc.

More generally, however, one might accept the need to analyze stochastic geometric data, using the tools of fractal analysis, without requiring that they conform to power law similarity relations. Non-power-law stochastic geometry similarity could be useful for analogous purposes, as was suggested by Miller and Dimotakis, who instead found log-normal similarity applicable in their analysis of turbulent jet fluid scalar fluctuation records.

Indeed, the log-normal distribution was originally proposed by Kolmogorov (1941b) to describe the self-similar fragmentation process resulting from successive rock crushings.

Acknowledgments

Many people have contributed, over some time, to the evolution of these ideas. Over the last few years, I would like to acknowledge the many discussions at Caltech with Paul Miller, with whom the experiments cited here were conducted, as well as Tony Leonard. Without wishing to imply endorsement, I would also like to acknowledge some recent discussions with Uriel Frisch and Larry Sirovich that spawned improvements in the exposition of the arguments presented here.

†P. Miller, private communication.

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